A. M. D. G.

BULLETIN

of the

American Association of Jesuit Scientists

Eastern States Division

(Founded 1922)

(for private circulation)

★

Published at

BOSTON COLLEGE

Chestnut Hill, Massachusetts

Vol. XXXI JANUARY, 1954 No. 2

Contents

Bulletin of the American Association of Jesuit Scientists

EASTERN STATES DIVISION

VOL. XXXI | JANUARY, 1954 | NO. 2

NOTICE TO AUTHORS

Manuscripts are to be submitted to associate editors of the appropriate section and *may* be submitted directly to the editor in chief. Clear manuscript, preferably typed, with wide margin to the left, with double spacing between lines, is desirable. Please try to follow the typographical style of the most recent issue of the BULLETIN. Line drawings should be submitted on Bristol board, or similar material, and done in India ink. Figure number should be written on this in pencil. Titles for drawings, with figure numbers, should be typed on a separate sheet. Please try to minimize footnotes. Appended references and bibliographies, clearly so marked, should be done in the style of the A.A.A.S. publication, *Science*.

Mathematics

ZERO IS NEITHER NOTHING NOR NOUGHT BUT A NUMBER[1]

JOSEPH T. CLARK, S.J.

Bellarmine College

> . . . Omitto infinita exempla in Aristotele, Platone, et eorum interpretibus illustrioribus, quae nulla ratione intelligi possunt sine mediocri scientiarum mathematicarum cognitione; immo propter earum ignorationem nonnulli philosophiae professores saepissime multos errores, eosque gravissimos, commiserunt, et (quod peius est) scriptis etiam mandarunt, quorum aliquos in medium proferre non esset difficile. Pari ratione oporteret praeceptores philosophiae callere disciplinas mathematicas, saltem mediocriter, ne in similes scopulos magna famae, quam Societas in litteris habet, iactura et dedecore incurrerent. . . . Ch. Clavius, *Monumenta paedagogica Societatis Iesu quae primam rationem studiorum praecessere* (Madrid: 1901), pp. 472-473.

---

[1] Text of an invitation lecture delivered at the Ninth Weston Science Colloquium, Weston College, Weston, Mass., November 22, 1953. This essay is respectfully dedicated to the revered memory of the late and much lamented Father Domhnall A. Steele, S.J., from whom I learned how to *learn* at last all the mathematics which I had previously studied and even successfully 'passed', and incalculably more besides. Father Steele was born in 1894 at Nantwich, Cheshire, England, and entered the Society of Jesus in 1915. In 1922 he received the degree of B.S. at University College, London. Ordained a priest at Roermond, the Netherlands, in 1925, he completed his theological studies there at Valkenburg the following year. From 1927 to 1935 Father Steele did graduate research in mathematics at the Universities of Goettingen and Bonn, Germany. Professor of mathematics at Bellarmine College, Oxfordshire, England, from 1935 to 1942 he went in the latter year to the Jesuit Residence in Edinburgh, Scotland where he devoted himself to mathematical research and writing until 1948. In that year he came to the United States as Professor of mathematics at Fordham University, New York City. Regarded by his professional colleagues in the field as a distinguished authority on the history of mathematics Father Steele wrote a treatise on the role of ruler and compass in Greek mathematical thought: *Über die Rolle von Zirkel und Lineal in der griechischen Mathematik* (Berlin: Verlag von Julius Springer 1936) which was acclaimed by Professor B. L. van der Waerden, internationally famous algebraist, as "the most authoritative work on the history of Greek mathematics." Father Steele

1.0 Zero Is not Nothing

Philosophers, classical philologists, and historians of physical science will remember that an incomplete study of the kinematics of bodies led Aristotle to conclude, erroneously as it happens, that for a given force or impulse, either naturally inherent or violently applied, the speed of a body in transit was in all cases inversely proportional to the density of the medium through which it moved. The gist of this Aristotelian principle may be expressed in the form of an equation: $s = k/d$, where s, the speed, is a function of d, the density of the medium, and k is a factor of proportionality. In the context of this pseudo-law Aristotle further raised the interesting theoretical question concerning the speed of a body in a vacuum, i.e. where $d = 0$. In reply Aristotle wrote, or so at least we are told in what is reputed to be the *best* available English translation of his *Physics*:

> Now there is no ratio in which the void is exceeded by body, as there is no ratio of 0 to a number. For if 4 exceeds 3 by 1, and 2 by more than 1, and 1 by still more than it exceeds 2, still there is no ratio by which it exceeds 0; for that which exceeds must be divisible into the excess + that which is exceeded, so that 4 will be what it exceeds 0 by + 0. . . . [2]

But I respectfully submit to a linguistically literate audience that this translation of the original Greek text is seriously misleading in its unqualified use of established mathematical symbols for numbers and operations throughout the passage. For Aristotle wrote here οὐδέν, which means literally 'no thing,' or better perhaps, 'nothing.'[3] But as it happens, zero is *not* nothing.

---

was a member of the Edinburgh and American Mathematical Societies, and in his later years in New York active as an associate of the *Friends of Scripta Mathematica* society, and a member of the interim executive committee for the foundation of the New York Metropolitan Chapter of the History of Science Society. Joint author with Father Walter Brugger, S.J., of the *Philosophisches Wörterbuch* to which he contributed definitive articles on more than sixty mathematicians and mathematical philosophers, Father Steele also wrote a spectacularly excellent introduction in historical vein for his translation of B. Bolzano's *Paradoxien des Unendlichen*. Father Steele's last publications, both articles and reviews, appeared by invitation in *Scripta Mathematica*. A glance at the annual catalogues of mathematics courses offered at Fordham University during the years of his tenure prove (1) that Father Steele never taught exactly the same course twice, and (2) that each succeeding year's syllabus was a definite advance over the preceding year's attainments. It is obvious that he grew with his work. Fellow members of the Mathematics Section of the Eastern States Division of the American Association of Jesuit Scientists will long remember Father Steele's enormously valuable contributions both of mathematical wisdom and rich humanist wit to its annual meetings. *Requiescat in pace.*

[2] Aristotle, *Physica* 4. 8. 215^b. The translation here reproduced is that of W. D. Ross and J. A. Smith, *The Works of Aristotle* (Oxford: Oxford University Press, 1908-1931), volume 2. The same version occurs in *The Basic Works of Aristotle*, edited by Richard McKeon (New York: Random House, Inc., 1941), p. 284.

[3] On this special point see the interesting paper by Carl B. Boyer, "An Early

2.0 Zero Is not Nought

Philosophers, Oriental philologists, and historians of astronomical science will perhaps also recall that in the positional notation of the Babylonian sexagesimal number system[4] it was desirable to possess an indicator for *empty* places in the sequence of enumeral symbols in order to avoid possible ambiguities in transcription and interpretation. But it is important to note that within a sexagesimal system such empty positions in a numerical expression occur far less frequently than in a decimal system. No such places, for example, occur at all for numbers less than sixty, and in only 59 cases for numbers less than 3600, as compared to 917 such occasions in a decimal system. Because therefore such notational ambiguities arose but seldom in routine calculations, the earlier Babylonians who first developed to some technical perfection a smooth positional system of numeration, felt no urgent need to invent an emptiness indicator. At some later time however, most likely in the Persian period, a conventional sign was adopted to mark whatever empty places did occur in a numerical expression constructed on the principle of positional notation and local value. Such Babylonian usage represents what is perhaps the earliest appearance of a symbol for *nought*.[5] But nought is *not* a number, no more than the space left vacant between the last letter of a preceding word and the first letter of the following word in conventional prose script is to be construed as the twenty-*seventh* letter of the English alphabet. For such a gap is not a letter, nor is nought a number, nor finally is zero nothing but nought.

3.0 Zero Is a Number

For zero is neither nothing nor nought but a *number*, as one may now proceed to show by rigorous and autonomous constructive methods.

3.1 The Point of First Constructive Departure[6]

I choose to think now, as an exercise of that splendid freedom

---

Reference to Division by Zero," *American Mathematical Monthly* 50 (1943) 487-491. Boyer remarks quite judiciously, and I am disposed to agree: "In this quotation it is quite evident that Aristotle had the arithmetical zero in mind. It is clearly distinguished from the philosophical void, and it is regarded as akin to, although not actually one of, the numbers." But to *be* a number, and to be *like* a number, are not by any means the same mathematical phenomenon.

[4] For those who are unfamiliar with the structure of a sexagesimal number system, it may be possible to convey an impression of it by equivalent transposition into the expressions of a decimal system. Thus for example the number, 162,032, would be written in appropriate symbols as: $45\ (60)^2 + 0\,(60) + 32$.

[5] For a sketch and survey of the reliable data and remaining doubts in the long and involved history of zero as nothing, as nought, as notation, as notion, and in less detail also as number, see Carl B. Boyer, "Zero: the Symbol, the Concept, the Number," *National Mathematics Magazine* 18 (1944) 1-8, and the copious standard reference literature there cited.

[6] Throughout this challenging essay I aim to be continually conscious of the

of creative intelligence with which mankind is endowed, of a set S and a relation R with one antecedent and one consequent. For purposes of successful and economical communication between us, I decide that such incidence between the antecedent p of R and the consequent q of R shall henceforward be symbolized in print as p/q, and read as 'p stroke q.' In the subsequent exploration of the necessarily implied logical consequences of such R over such S the letters a, b, c, are now chosen to be the names of *given* elements of S, while the letters x, y, z are selected to serve as names for elements of S hypothetically supposed and tentatively entertained until affiliated as members of S by demonstrative proof. Thinking thus of a set S and such a relation R over S, I further choose to investigate now any instance *whatever* in which the following seven conditions happen to be the case for S:

(1) there is in S some one element u which is such as described hereinafter from (2) through (7), and

(2) for every a in S there is at least some one or other element x in S such that identically in S a/x, and

(3) for every a in S such that $a \neq u$, there is at least some one or other element x in S such that identically in S x/a, and

(4) it is not the case that there exists in S an element x such that x/u, and

(5) for every a in S, if there is a suitable x at all, then there is at most one such x such that x/a, and

(6) for every a in S there is at most some one or other element x in S such that identically in S a/x, and finally

(7) subsets of S comprising u and with v also its consequent v' have empty complements in S.(7)

---

widely *diverse* interests and capacities of the general audience. I plan to meet this challenge of irreducible heterogeneity by the following device and here interrupt development of the theme to acquaint cooperative readers with the details of its execution: (1) henceforward the *text* as such purports to be intelligible to the high-school graduate who can read and understand idiomatic English; (2) footnote superscript indices *without* parentheses are hints to the *more* capable or to the more *interested* to pursue some points a little more deeply than is necessary for mere comprehension, and (3) footnote superscript indices *within* parentheses are pointers to materials designed to satisfy the legitimate demands for justification or just further curiosity on the part of the technically trained mathematical members of the audience.

(7) No independence is claimed for statements (1) through (7) above. On the contrary (3) is entailed by (2) and (7) jointly taken. Note in general that justificatory appeals to (7) regularly fall into two correlated and recurrent stages: (a) a *beginning* stage, henceforward designated as *Stage B*, to prove the assertion in question true of the special element u in S, and (b) a *transference* stage, henceforward denoted as *Stage T*, to prove the same assertion true of v' when true of v. Hereby v' thus denotes that element in S which in virtue of (2) and (6) is the solitary consequent to v as antecedent. The present proof thus proceeds as follows: select from S a subset S^* such that it contains u and all potential consequents. *Stage B*: u belongs to S^* and thus also to S by explicit choice. *Stage T*: when v belongs to S^* and thus also to S, v' exists in S by (2) and is thereby guaranteed to be a potential (because in fact an actual) consequent, and thus is enfranchised as a member of S^* in S. Hence (3) is entailed by both (2) and (7). But six of the

3.2 Construction of a First New Relation over S

I continue to think of the same set S as described in statements (1) through (7) of 3.1, and now propose (i) to construct over S a function[8] $f(x,y)$, such that always throughout S it is the case that

$$(1)\ f(x,u) = x' \qquad \text{and} \qquad (2)\ f(x,y') = f(x,y)'$$

and (ii) also to show that there is over S only a *single* such function. It is more convenient in exposition to reverse the order of (i) and (ii) and thus to show (ii) first as

Theorem 1: for fixed c in S there is *at most* one manner of effecting

---

relatively primitive propositions stated above, (1), (2), (4), (5), (6), (7), are in fact such that each alternative selection of five can be so interpreted that the sixth in each case turns out to be false under the corresponding interpretation. Hence these six are logically independent propositions. For the general role of independence in axiomatic system structure see my "Contemporary Science and Deductive Methodology," *Proceedings of the American Catholic Philosophical Association* 26 (1952) 94-131, and the standard reference literature there listed.

[8] It is to be noted that the precise and accurate concept of functionality contains two constitutive elements: *relationship* and *dependency*. It may be wise to open a condensed and elementary exposition of the matter by a fundamental interpretation of the symbol: '$f(x)$.' This conventional symbol which is correctly read, not 'eff-ex' but 'eff-*of*-ex,' presupposes two clear conditions. The first is this: (1) *a well-defined class of mathematical entities* which may here be called K. The letter 'x' may next be chosen to *name* any one entity from the class K. Such an entity of such a class is usually called on understandable historical (but deplorable logical) grounds by the systematically misleading title of 'variable.' But the unsophisticated reader should be warned that exactly no kind of change in x is involved, nor is any kind of change in x at all possible. For each individual mathematical object, such as a number, is an immutable entity and can under no circumstances whatever neither suddenly nor gradually be transformed into another mathematical object, such as another and distinct number. But what may occur is this: a responsible mathematical intelligence may have good reasons to consider a succession of different entities, such as different numbers, arranged by previous constructive definition in some one or other regular sequence. 'Variability' therefore in the context of functionality does *not* mean nor does it imply 'changeableness,' but rather 'anyness-over-a-specified-set.' The second prerequisite is this: (2) *a well-defined mathematical relation* which may here be called 'f' and is such that given any *one* member x of the class K, just one mathematical entity, which we may conveniently call 'y,' stands in the relation f to the entity x. The entire situation is neatly symbolized as: $y = f(x)$. Observe that the set of all such y itself forms a well-defined class which depends upon the class K *and* upon the relation f, and may here be called 'M.' The mathematically free choice of some one or other x from K leads through the link of the relation f to the *bound* choice of the corresponding y from M. It is conventional therefore to refer to such x as the *independent* variable and to such y as the *dependent* variable. The relation f may be constituted in several different ways: by a computational formula, by a verbal proposition, or perhaps again as a solution of a problem known on other and independent grounds to possess exactly one unique solution. A second such relation, if such there be, may be called '$g(x)$,' a third '$h(x)$,' and so forth. Moreover to state that $f(x) = g(x)$ *identically* in x means that whatever be the choice of x from K, one and the same y from M will be found to stand to x in both the relation f and in the relation g simultaneously. Similarly but with further structural complications there may well exist functions such as $f(x,y)$, $g(x,y)$, $f(x,y,z)$, or $g(x,y,z)$ of two, three, or n variables. Such polyadic functions are based in logical terms upon two-one, three-one, or in general many-one relations.

$(3)\ f(c,u) = c'$ and $(4)\ f(c,y') = f(c,y)'$.[9]

and then (i) second as

Theorem 2: for fixed c there is *at least* one manner of effecting $(3)\ f(c,u) = c'$ and $(4)\ f(c,y') = f(c,y)'$.[10]

3.3 Significant Properties of Such $f(x,y)$ over S

The function $f(x,y)$ over S which 3.2 shows to be both existent and unique, also possesses the following significant logical properties:

Theorem 3: $f[f(x,y),z] = f[x,f(y,z)]$.[11]

Theorem 4: $f(x,y) = f(y,x)$.[12]

Theorem 5: $f(x,y) \neq x$ and $f(x,y) \neq y$ for every x and y in S.[13]

Theorem 6: if $y \neq z$, then $f(x,y) \neq f(x,z)$.[14]

Theorem 7: for given x and y, exactly three strictly exclusive alternatives hold within S:

---

(9) Proof: suppose *also* over S $g(c,u) = c'$ and $g(c,y') = g(c,y)'$. Select then the subset S^* of S containing all v for which $f(c,v) = g(c,v)$. *Stage B*: the common value $f(c,u) = g(c,u) = c'$ puts u itself into S^*. *Stage T*: $f(c,v') = f(c,v)'$ by the defined character of f, and $f(c,v') = g(c,v)'$ by (6) in 3.1, and finally $f(c,v') = g(c,v')$ by the supposed character of g. Whence it follows that v' itself is enfranchised for admission into S^*. Hence *if* there is such a function f over S, it is *unique* over S.

(10) Proof: select the subset S^* of those x in S which do secure (3) and (4) above. *Stage B*: the exploratory choice $f(u,y) = y'$ does verify (3) when $y = u$, and does furthermore verify (4) for *every* y insofar as then $f(u,y') = y'' = f(u,y)'$. *Stage T*: when x verifies (3) and (4) one must again make such a choice of x' that there is again a verification. Choose $f(x',y) = f(x,y)'$. Then again (3) is verified when $y = u$ because $f(x',u) = f(x,u') = x''$, and (4) is verified for *every* y insofar as $f(x',y') = f(x,y')' = f(x,y)'' = f(x',y)'$.

(11) Proof: keep x and y fixed at otherwise arbitrary elements of S. Select next the subset S^* of S which contains all those z for which theorem 3 is valid. Then *Stage B*: S^* contains u since $f[f(x,y),u] = f(x,y)' = f(x,y') = f[x,f(y,u)]$. *Stage T*: when S^* contains z, it is the case that $f[f(x,y),z] = f[x,f(y,z)]$ and hence $f[f(x,y),z'] = f[f(x,y),z]' = f[x,f(y,z)]' = f[x,f(y,z)']$ $= f[x,f(y,z')]$. Comparison of the extremes proves that v' is qualified to enter into S^* of S.

(12) Proof: keep y fixed. Select the subset S^* of S which contains all those x which do verify theorem (4). *Stage B*: theorem (3) guarantees already that $f(u,y) = y'$ and $f(y,u) = y'$, so that here at least $f(u,y) = f(y,u)$. *Stage T*: in any case $f(x',y) = f(x,y)'$ by theorem 3, and $f(x,y)' = f(y,x)'$ by (6) in 3.1, so soon as S^* contains x, and $f(y,x)' = f(y,x')$ by the reverse application of theorem 3.

(13) Proof: keep x fixed. Select next the subset S^* of S that contains those y which verify theorem 5. *Stage B*: never is $u = x'$ because (4) in 3.1 proscribes it. But $x' = f(x,u)$. Therefore never is it the case that $u = f(x,u)$. *Stage T*: given that $y = f(x,y)$ is *false* over S, then $y' = f(x,y)'$ is also false over S by (5) in 3.1. Moreover since $f(x,y)' = f(x,y')$, neither is it ever the case that in S $y' = f(x,y')$.

(14) Proof: choose distinct y and z and hold each fixed. Next select the subset S^* of S that contains those x which do verify theorem 6. *Stage B*: $y \neq z$ entails $y' \neq z'$ by (5) in 3.1. Hence by definition $f(u,y) \neq f(u,z)$. *Stage T*: given about x that $f(x,y) \neq f(x,z)$, then $f(x,y)' \neq f(x,z)'$ by (5) in 3.1, and therefore it follows that $f(x',y) \neq f(x',z)$.

either Case 1: $x = y$,
or Case 2: $x = f(y,s)$ for some s unique by theorem 6,
or Case 3: $y = f(x,t)$ for some t unique by theorem 6.

For although the three cases are mutually exclusive,(15) all cannot be false in S.(16)

3.4 Construction of a Second New Relation over S

Theorem 7 in 3.3 furthermore lays down the foundations for the rigorous construction of a second new binary relation over S, such that that whenever $x \neq y$, a condition which thus excludes Case 1 under theorem 7 in 3.3, then x/y holds in Case 2 and y/x holds in Case 3, either one of which must in every other instance be the case in S. It follows immediately that such constructed R_2 over S is *alioalternative*. Furthermore such R_2 over S is also *transitive* because

(a) x/y exactly when $x = f(y,s)$ for some s,
(b) y/z exactly when $y = f(z,t)$ for some t, and
(c) there then holds by reason of theorem 3 in 3.3 associativity

whereby $x = f[f(z,t),s] = f[z,f(t,s)]$, which is sufficient to establish x/z through the existence in S of this $f(t,s)$. Hence precisely as alioalternative and transitive, this new and constructed R_2 over S is by definition *serial*.

3.5 Construction of a Third New Relation over S

There exists over S one and only one function $g(x,y)$ such that both (i) $g(x,u) = u$ and (ii) $g(x,y') = f[g(x,y),x]$. Such R_3 over S is likewise associative and commutative or symmetric, and also in

---

(15) Proof: Cases (2) and (3) are incompatible because if both held in S, then it would be the case that $x = f(y,s) = f[f(x,t),s] = f[x,f(t,s)]$. But the extremes contradict theorem 5. Cases (3) and (1), like (1) and (2), are also incompatible because they too similarly contradict theorem 5 explicitly.

(16) Proof: keep x fixed. Select the subset S^* of S which contains all those y for which some one or other of the three alternative cases holds. *Stage B*: when $y = u$ and $x = u$, then $y = x$ and Case (1) is given immediately; then $y = u$ and $x \neq u$, then $x = s'$ by (3) in 3.1, whence $x = f(u,s) = f(y,s)$ which is an instance of Case (2). *Stage T*: let us proceed by distinct parts. Given that y lies in some one or other of the three Cases, it is required to show that thereby y' also lies in some one or other of the three Cases. The following set of four such possible connections is relevant:

(a) if y is in Case (1), then y' is in Case (3),
(b) if y is in Case (2), with $s = u$, then y' is in Case (1),
(c) if y is in Case (2), with $s \neq u$, then y' is in Case (2), and
(d) if y is in Case (3), then y' is in Case (3).

If hypothesis (a) holds, then $y' = f(y,u) = f(x,u)$ and therefore y' belongs to Case (3) with u in the role of t. If (b) holds then $x = f(y,u) = y'$, and y' falls into Case (1). If (c) holds, then $s = r' = f(u,r)$ by (3) in 3.1, so that $x = f(y,s) = f[y,f(u,r)] = f[f(y,u),r] = f(y',r)$, which equips y' for Case (2). Finally if hypothesis (d) holds, when $y = f(x,t)$, then $y' = f(x,t)'$ by (6) in 3.1, whereby y' is already enfranchised under Case (3) with present t' as surrogate for standard t.

conjunction with $f(x,y)$ distributive in the sense that identically in x, y, z it is the case that $g[x,f(y,z)] = f[g(x,y),g(x,z)]$.

3.51 First Retrospective Recapitulation before Advance

Section 3.1 which served as our point of departure, provided us with a set S and a binary relation R over S, such that

(1) S contained u as hereinafter described, and
(2) in S a/x had at least one solution, and
(6) one solution at most, now designated as a', and
(3) unless $a = u$, at least one solution existed for x/a, and
(5) one solution at most, but
(4) x/u had exactly no solution in S; finally
(7) subsets of S comprising u and with v also v' had empty complements in S.

Thereupon 3.2 established the existence over S of a first new constructed and *unique* binary relation R_1, such that in a functional expression of R_1 (i) $f(x,u) = x'$ and (ii) $f(x,y') = f(x,y)'$. Section 3.3 then explored the logical consequences of R_1 over S and demonstrated that (i) $f(x,y)$ is both associative and symmetric, (ii) never is $f(x,y) = x$ or $f(x,y) = y$, (iii) $y \neq z$ entails $f(x,y) \neq f(x,z)$, and (iv) strictly either $x = y$, or $x = f(y,s)$ or $y = f(x,t)$ for some suitable s or t. Next 3.4 undertook to establish a second new binary relation R_2 over S, such that when $x \neq y$, then x/y holds if $x = f(y,s)$, but y/x obtains whenever $y = f(x,t)$. Section 3.4 concluded this exposition by showing that because R_2 over S is both alioalternative and transitive, it is therefore serial over S. Finally section 3.5 undertook to establish the existence of a third new constructed and *unique* relation R_3 over S such that (i) $g(x,u) = x$, and (ii) $g(x,y') = f[g(x,y),x]$, so that (iii) R_3 is both associative, commutative or symmetric, and distributive with R_1 in the sense that identically in x, y, z it is the case that $g[x,f(y,z)] = f[g(x,y),g(x,z)]$. It is essential to our present purposes to realize at this point that each and every such construction, each and every such existence, each and every such uniqueness, each and every such demonstration was achieved solely from the data of the relatively primitive propositions (1) through (7) of 3.1 together with the resources of the logic of relations *antecedently to and independently of any recognition of exemplification or occurrence of some such set S within or without mathematics.* Moreover while the construction from 3.1 through 3.5 resembles somewhat the pioneer work of Peano (1858-1932)[17] and less so that of Frege (1848-1925),[18] there are significant differences from both Peano[(19)] and Frege.[(20)]

---

[17] Giuseppe Peano, [1] *Arithmetices principia nova methodo exposita* (Torino: 1899); [2] "Sul concetto di numero," *Rivista di mathematica* (Torino) 1 (1891) 87-102, 256-267; [3] *Formulaire de mathématiques* 1 (Torino: 1895), 2.1 (Torino: 1897), 2.2 (Torino: 1898), 2.3 (Torino: 1899), 3 (Paris: 1901), 4 (Torino: 1902-1903), 5 (Torino: 1905-1908).

[18] Gottlob Frege, [1] *Die Grundlagen der Arithmetik, eine logisch-mathematische*

4.0 The Abstract but not Abstracted Character of S

Up to this point therefore the *nature* of the elements of S and of the respective relations over S have consistently been of exactly no concern whatever to the successive processes of autonomous construction. For conscious attention has been exclusively centered upon the merely *logical* (and *not* ontological) character of the relations over S, and steady concern exhibited for the elements of S *only* insofar as they functioned respectively and alternately as the antecedents or the consequents of the several relations over S. Hence the elements of S are now known to us *only* as the relata of the relations over S and correspondingly the relations over S are now known to us *only* as the links that correlate the elements of S. Nor is such knowledge in any way deficient because the elements and relations of S are *only* the elements and

---

Untersuchung über den Begriff der Zahl (Breslau: 1884), and [2] *Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet* 1 (Jena: 1893), 2 (Jena: 1903). Item [1] was reprinted at Breslau in 1934 and with interpaginal English translation by J. L. Austin at New York, Philosophical Library, 1950.

(19) Peano's mathematical researches sought a set of relatively primitive propositions upon which to establish arithmetical science. Exactly in the manner of Euclid and Euclid's own authorities, *Peano analyzed backwards from the middle* of the scientific arithmetic of his time. He therefore set down as an adequate foundation for the structure of all fundamental arithmetic three undefined primitive terms and five unproved propositions, as follows: (a) undefined terms: *unità, numero, successivo;* (b) unproved propositions: (1) *l' unità è un numero,* (2) *il segno + messo dopo un numero produce un numero,* (3) *se A e B sono due numeri, e se i loro successivi sono eguali, anche essi sono eguali,* (4) *l' unità non segue alcun numero,* (5) *se S è una classe, contenente l' unità, e se la classe formata dai successivi di S è contenuta in S, allora ogni numero è contenuto nella classe S.* Comparison with sections 3.1 to 3.5 in the text will show that the entire present treatment is a *more refined relational distillate* of the pioneer work of Peano. The principal differences in this connection are two: (1) preference is here shown to the term 'consequent,' taken from the abstract logic of relations, rather than to the too concrete term of Peano 'successor'; (2) sections 3.2 and 3.5 take cognizance of the necessity to prove the *uniqueness* of $f(x,y)$ and $g(x,y)$ over S, as taught by Grandjot and Kalmàr and neatly summarized by E. Landau, *Foundations of Analysis* (New York: Chelsea Publishing Co., 1950). Moreover the reader should note that the above construction does not proceed like Peano in other versions from a stipulated 0 as appendage to the natural numbers, but awaits its later and independent construction.

(20) Frege took capacity for biunivocal correspondence or cardinal similarity as an equivalence relation between sets and construed each separate resulting class as a cardinal or natural number. It is essential, but sometimes difficult for the tyro, to understand that for Frege the class *was* the number. In order to obtain sums, Frege first proves that from two given classes of sets some member set may be so chosen from the former and some member set from the latter that the two selected sets are disjoint; thereupon one constructs a new set by way of a logical sum function: 'belonging to the one *or* to the other;' finally this new set is shown to lie in the same class *no matter what representatives from the original classes of sets are chosen.* Thus this new class is the additive relatum of the classes first taken. In order to obtain products, Frege chooses representatives as above and then constructs by a logical product function a new set consisting of all *couples* such that their prior members are in the prior set and their posterior members in the posterior set. Since this class to which the set of couples belongs, depends only on the original classes *no matter what representatives are chosen therefrom,* it is a function of them, called the product function.

relations of S. At no time therefore was the fatuous attempt made to explain what the elements of S are supposed to be because the elements of S are *not* supposed to be anything other than the elements of S. Similarly at no time was an effort made to explain what the relational system of S was all about, simply because S is identically S and is not about anything else whatever other than S.

4.1 A Relevant Mathematical Interpretation of S

But it is a fact that the relational morphology of S which is open to very many and widely diverse kinds of interpretation, can at least be so interpreted within mathematics that its logical structure is seen to be realized in a very familiar domain, that of the natural numbers. For one may interpret the member elements of S as the natural numbers in such wise that the special element u is identified as the number 1, and the remaining natural numbers specified on the plan that $1' = 2$, $2' = 3$, $3' = 4$, etc., as was first suggested by Leibniz. Moreover $f(x,y)$ is successfully interpretable as $x + y$ and $g(x,y)$ as xy. Finally the initial binary relation x/y may be construed as $x > y$ and conversely by convention $y < x$. Thereupon the following arithmetical propositions cease to be mere arbitrary rules of arithmetical computation which somehow just happen to work well, or even baseless conventions accepted by a polite gentlemen's agreement, and attain the desirable status of rigorously demonstrated theorems:

(1) $x + (y + z) = (x + y) + z$,
(2) $x + y = y + x$,
(3) strictly either $x = y$, or $x > y$, or $x < y$, and
(4) $y \neq z$ entails $x + y \neq x + z$.

In this manner the entire arithmetic of the natural numbers can be constructed from a competently declared set of relatively primitive propositions, suitably interpreted. There is therefore now at our disposal for future purposes the entire set of natural numbers but *only* as yet the set of natural numbers: 1, 2, 3, etc.[21]

---

[21] For future purposes certain attributes of natural numbers will be useful and are here listed together for ease in reference without further proof:

(1) $a + b$ in the set and unique,
(2) $a + (b + c) = (a + b) + c$,
(3) $a + b = b + a$,
(4) $a + x = b$ but not always solvable in the set,
(5) ab in the set and unique,
(6) $ab.c = a.bc$,
(7) $ab = ba$,
(8) $a(b + c) = ab + ac$,
(9) $x + y \neq x$ for all x and for all y,
(10) $a + x = a + y$ entails $x = y$.

Attention is called particularly to items (10) and (4). Item (10) states that *if* subtraction is possible at all, *then* it is unique. Item (4) announces that subtraction is *not* always possible in the set.

4.2 Second Retrospective Recapitulation before Advance

We thus stand now at a firm place in intelligence whence we may with a rigorously certified logical license descend along the autonomously constructed scale of the natural numbers from whatever member you please down to . . . 5, 4, 3, 2, 1 *STOP!* But we lack at this stage the right to proceed in good conscience from whatever member you please to whatever member you please along the integer scale . . . , +5, +4, +3, +2, +1, 0, —1, —2, —3, —4, —5, ... And in particular we still lack the pivotal *number* zero. It is furthermore already clear from section 1.0 that although we all may, as a popular ballad puts it, 'have plenty o' nothin',' yet zero is *not* nothing, and therefore nothing will not supply to us the number zero. Moreover, although it be little enough trouble to invent a suitable symbol for nought as an emptiness indicator in a positional number system, still zero is *not* nought either and hence nought is no substitute for the legitimate number zero. And at least one good reason for both of these failures is this: the number one and nothing, and the number one and nought, have exactly no defined sum at all, whereas the number one and the *number* zero do have a unique defined sum, i.e., the number 1. At this point then there may perhaps arise in the mind of the impatient listener the revealing question: "But why in the world should one be seriously concerned correctly to define zero, since it would appear in effect that the entire enterprise is nothing else but just another case of much ado about nothing?" Some may therefore be inclined to rest content with the unreflective and crassly pragmatic use of conventional symbols, rules, and recipes that just happen to work well in computational experience. But such a question and such an attitude set a grave crisis for the truly mathematical conscience and at one stroke separate the men from the boys in the world of honest intelligence. By way of reply one may then heed the explicit ideals of Plato, the sharp criteria of Gauss, and the pathetic sighs of our Father Clavius. In the *Cratylus* 439B, for example, Plato remarks as representative spokesman for conscientious mathematicians: "We have to rest content with the confession that our study and research must be conducted, not on the strength of names, but on the strength of things themselves."[22] And the great Gauss in similar authentic mathematical vein declares with reference to a certain theorem of Waring $[p/1 + (p-1)\,!]$:

> . . . Theorema hoc elegans primum a celeberrimo Waring [1734-1798] est prolatum, armigeroque Wilson [1741-1793] adscriptum . . . Sed neuter demonstrare potuit et celeberrimus Waring fatetur demonstrationem eo difficiliorem videri, quod nulla notatio fingi possit, quae numerum primum exprimat. At nostro quidem

---

[22] Ἀγαπητὸν δὲ καὶ τοῦτο ὁμολογήσασθαι, ὅτι οὐκ ἐξ ὀνομάτων, ἀλλὰ πολὺ μᾶλλον αὐτὰ ἐξ αὐτῶν καὶ μαθητέον καὶ ζητητέον ἢ ἐκ τῶν ὀνομάτων.

iudicio huiusmodi veritates ex n o t i o n i b u s potius quam ex n o t a t i o n i b u s hauriri debebant.[23]

It is apparent how closely the sentiments of both Plato and Gauss coincide with respect to the ideally rigorous criteria that rule the mathematical enterprise. There is finally perhaps no more pathetic passage in all humanist literature than the poignant anxiety of a very candid Clavius:

> . . . Causa autem huius rei in multiplicatione numerorum cossicorum [e.g., x, y, x^2, y^2, etc.], et signorum + et —, reicienda videtur; et debilitas ingenii humani accusanda, quod capere non potest, quo pacto id verum esse possit. Neque enim de ratione praedictae multiplicationis dubitandum est, cum illa per multa exempla sit confirmata.[24]

The challenge to intelligence set by Clavius is clear. Either the Rule-of-Signs in multiplication is simply a brute enigma, a mathematical fact inscrutable to human reason, or else the inventive ingenuity of creative intelligence can construct the existence of the negative, null, and positive integers out of the raw materials of the natural numbers and thus transform the Rule-of-Signs from an unintelligible convention into a demonstrated theorem. Let us accept the challenge and try our hand at a solution.

5.1 The Point of Second Constructive Departure

When the following differences exist at all in the domain of the natural numbers,[25] we already know that when a, B, and A, b are pairs of suitable naturals

(1) $(a — A) = (b — B)$ if and only if $a + B = A + b$,
(2) $(a — A) + (b — B) = (a + b) — (A + B)$, and
(3) $(a — A)\ (b — B) = (ab + AB) — (aB + bA)$.

Such pairs of natural numbers with such pair properties first suggest an attempt to work out a constructive extension of the number domain by using *the set of all couples of naturals*: $[a,A]$, $[b,B]$, $[s,S]$, $[p,P]$, and then to establish over them exploratory (hence here parenthesized plus and cross) additive and multiplicative relations, so that

(i) $[a,A]\ (+)\ [b,B] = [a + b,\ A + B]$, and
(ii) $[a,A]\ (\times)\ [b,B] = [ab + AB,\ aB + bA]$,

and finally to examine how far, if at all, the newly constructed relations over *couples* of naturals are analogous and thus comparable to the existing addition and the existing multiplication relations over the existing individual natural numbers. Close inspection discloses that (i) and (ii) are indeed quite similar to the original addition and multi-

[23] *Disquisitiones arithmeticae*, 76. See the *Werke*, I, p. 60.
[24] *Algebra*, 6. See the *Werke*, II, p. 17 of the *Algebra* pagination.
[25] Consult footnote 21 of this essay and compare therein especially items (10) and (4).

plication *but actually yield no increased power of substraction.* For example: $[7,5] + [x,X] = [3,2]$ would thus be an identity that requires both that $7 + x = 3$ and that $5 + X = 2$ among the natural numbers. But the latter two identities are there impossible by default of such an x and such an X. To proceed further therefore along this road *in the same direction* would be to come to the dead end of a blind alley.

5.2 A New Direction from the Second Departure

But one may perhaps successfully salvage the original hunch derived from an inspection of a certain discernible behavior of the natural numbers, and continue to explore the constructive possibilities of working with couples of *natural* numbers from a *different* approach and with a new and *different* strategy. In the designedly empty matrix diagram below, write in *at random* at most one entry each for every representative sample couple of the natural numbers selected for reasons of graphical simplicity *only* from 1 to 9 inclusive, such as (1,9), (8,2), (2,8), (3,3), etc.

Diagram I

,	,	,	,	,	,	,	,	,
,	,	,	,	,	,	,	,	,
,	,	,	,	,	,	,	,	,
,	,	,	,	,	,	,	,	,
,	,	,	,	,	,	,	,	,
,	,	,	,	,	,	,	,	,
,	,	,	,	,	,	,	,	,
,	,	,	,	,	,	,	,	,
,	,	,	,	,	,	,	,	,

In the completed diagram above let us now examine together this exhaustive sample array of appropriate couples of naturals from 1 to 9. The first observation is this: either the chart exhibits no recognizable order at all, or at most an incomplete and imperfect pattern, and certainly in no case a consciously constructed and rigorously controlled one. The second and significant observation is this: each and every couple of naturals verifies at most but one or another of the following three alternatives:

(1) either there is *no* serial difference between the prior and posterior members of a couple, such as (3,3),

(2) or the prior member is serially later than the posterior member, such as (6,2),

(3) or the prior member is serially earlier than the posterior member, such as (4,5).

Now to the mathematically sensitive conscience these three strict alternatives which thus obtain over couples of naturals supply a sound basis for the extension of serial order beyond the natural number domain, provided we have the wits to devise an effective logical technique to exploit and control it for constructive purposes.

5.3 Constructive Procedure for a Diagonal Calculus

In order to exploit such given but latent possibilities for the construction of a richer and more extensive realm of serial order out of the raw materials of couples of natural numbers, let us attempt to classify and to distribute on appropriate *diagonals* all the couples of Diagram I according to the criterion that the couples $[a,A]$ and $[x,X]$ are to be admitted to the same diagonal if and only if $a + X = A + x$, so that for example on this criterion the couples (1,1), (1,9), (9,1), and (9,9) qualify for the positions allotted to them in Diagram II below. On the same criterion and after the same manner fill in all the remaining places in Diagram II with the appropriate corresponding sample couples from the array in Diagram I.

Hence by classifying couples of natural numbers according to the criterion that two such couples fall into the same class and on the same diagonal if and only if the above relation holds, and otherwise into different classes and diagonals, we make Diagram II effectively exhibit *individual classes* of couples of naturals as *individual diagonals.*[26] In order now to consolidate this present advance and to facilitate communication for further progress, let us agree to write

(1) *square brackets*, $[a,A]$ for individual *couples* of naturals,

(2) *round parentheses*, (a,A) for the *diagonal* containing all such $[a,A]$, and

(3) *three horizontal strokes*, $\equiv$, to mean 'is a *diagonal*-mate or class-mate of.' Hence by these conventions $a + B = b + A$, $[a,A] \equiv [b,B]$, and $(a,A) = (b,B)$ are synonymous.

5.4 The Construction of Diagonal Quasi-sum

Among the natural numbers it is the case that $a + P = A + p$ and $b + Q = B + q$ jointly entail that $(a + b) + (P + Q) =$

[26] In a perspective broader than our present one, it is a problem in modern logic to determine precisely when a one-to-one relation does in this manner effectively produce an exact partition of a set into distinct and non-overlapping subclasses. But in the present simple and logically privileged case the above relation is seen immediately to achieve this result.

Diagram II

(1,1)	,	,	,	,	,	,	,	(1,9)
,	,	,	,	,	,	,	,	,
,	,	,	,	,	,	,	,	,
,	,	,	,	,	,	,	,	,
,	,	,	,	,	,	,	,	,
,	,	,	,	,	,	,	,	,
,	,	,	,	,	,	,	,	,
,	,	,	,	,	,	,	,	,
(9,1)	,	,	,	,	,	,	,	(9,9)

$(A + B) + (p + q)$. Hence on the criterion of the present class-mateship condition $(a,A) = (p,P)$ and $(b,B) = (q,Q)$ jointly entail $(a + b, A + B) = (p + q, P + Q)$. But this is to say in other words that the *diagonal* to which $[a + b, A + B]$ belongs is determined *solely* by the *diagonals* to which $[a,A]$ and $[b,B]$ severally belong, *and not at all by the actual choice of a representative couple from any diagonal.* But this is to say in still other words that the actual *diagonals*, as newly constructed mathematical entities, enter into a relation *induced by a previous and existent relation between diagonal constituents.* Let us therefore provisionally be permitted to call $(a + b, A + B)$ the *quasi-sum* of (a,A) and (b,B), *sum* because this designation will soon be justified, and *quasi* because such necessary justification is yet to come.

5.5 The Construction of Diagonal Quasi-product

The conditions set in 5.3 under which the individual duplexes $[a,A] \equiv [p,P]$ and $[b,B] \equiv [q,Q]$ are respectively $a + P = A + p$ and $b + Q = q + B$, may furthermore be multiplied on both sides, the former by b and B, the latter by p and P, so that

(1) $(a + P)b = (A + p)b$,
(2) $(A + p)B = (a + P)B$
(3) $(b + Q)p = (q + B)p$, and
(4) $(q + B)P = (b + Q)P$.

By adding as one may among naturals and cancelling as one also may among naturals, it turns out that

$$(5)\ ab + AB \ + \ pQ + qP = aB + ba \ + \ pq + PQ.$$

But the reader will observe that the identity in (5) states explicitly and exactly both the necessary and sufficient conditions that

$$(6)\ (ab + AB, aB + Ab) = (pq + PQ, pQ + Pq).$$

But what (6) says in the precise notational symbolism adopted in 5.3 is that the *diagonal* to which $[ab + AB, aB + Ab]$ belongs is determined *solely* by the *diagonals* to which $[a,A]$ and $[b,B]$ severally belong, *and not at all by the actual choice of a representative couple from any diagonal.* In other words the actual *diagonals*, as newly constructed mathematical entities, enter into a relation *induced by a previous and existent relation between diagonal constituents.* Let us therefore provisionally be permitted to call $(ab + AB, aB + Ab)$ the *quasi-product* of (a,A) and (b,B), *product* because this term will soon be justified, and *quasi* because such justification is yet to come.

5.6 Third Retrospective Recapitulation before Advance

It is important now for purposes of future progress to pause at this point, to reconnoitre the traversed terrain carefully, and to determine exactly what is the anatomical structure of this newly constructed *diagonal calculus.* In order to make such analysis effective, it is necessary to introduce and to describe briefly the mathematical concepts of *group* and *ring.*

The notion of *group* is fundamental in all branches of mathematics. It acts like a staining fluid in biological research. For so soon as the notion of group is injected into any mathematical theory, it instantly reveals intuitively unsuspected but centrally important structural details. By definition a set which is argument-closed and value-closed *with respect to some one or other associative two-one relation* is called a group. If therefore some set S is to qualify as a group *relative to f*, the following four and no fewer than four conditions must jointly be met:

(1) S contains, if x and y, then also a *unique* z with $z = f(x,y)$,
(2) S contains, if x and z, then also a *unique* y with $z = f(x,y)$,
(3) S contains, if y and z, then also a *unique* x with $z = f(x,y)$,
and finally
(4) $f(x,y)$ is associative always.[27]

[27] Even introductory manuals on the theory of groups do not denote the f-relatum of x and y by the unambiguous *functional* notation '$f(x,y)$,' but by a conventionally standardized use of the product symbol 'xy,' whereby 'xy' names that element of the group which stands in the group relation to x in the first place and to y in the second place. Such idiomatic use of the product notation in group theory soon becomes convenient and harmless, but it may be misleading initially to

Whenever the above conditions (1) through (4) are satisfied for any set S, then it is possible to prove as a further demonstrated theorem that every such group contains just one element, here called its *neutrum,* and symbolized by e, such that $x = f(x,e) = f(e,x)$ for every x in the group.[28]

Groups are therefore closed with respect to *one* function, called above $f(x,y)$. *Rings* are groups with respect to *two* functions, hereafter called $S(x,y)$ and $P(x,y)$. By definition therefore a set R is called a ring *relative to the functions* S and P if and only if it meets the following eight and no fewer than eight conditions distributed into three distinct classes:

(A) with respect to S alone:

(1) R contains, if a and b, then a *unique* $S(a,b)$,

(2) S is associative,

(3) S is also commutative,

(4) R contains a *unique* solution for x such that $S(a,x) = b$,

(B) with respect to P alone:

(5) R contains, if a and b, then a *unique* $P(a,b)$,

(6) P is associative,

(C) with respect to S and P jointly:

(7) S and P are left distributive, so that $P[a,S(b,c)] = S[P(a,b),P(a,c)]$, and

(8) S and P are right distributive, so that $P[S(b,c),a] = S[P(b,a),P(c,a)]$.[(29)]

Comparison between items (1) to (4) of the conditions for a group and items (1) to (4) of the conditions for a ring discloses immediately that with respect to their S-function rings are commutative groups, and as groups fall therefore under all generalized group

---

the layman. For it conceals the *non*-trivial character of associativity and renders the occurrence of non-commutative groups a mathematical phenomenon that seems more strange than need be. The beginner may reasonably be disconcerted to find in such groups that $xy \neq yx$, where he would presumably not be at all dismayed to discover that $f(x,y) \neq f(y,x)$.

[28] In the product notation conventional in group theory and mentioned in footnote 27, the present *neutrum* is called after the neutrum for multiplication 'unit element' of the group, or sometimes 'identical element,' or again sometimes 'principal element.' The present reasons for preferring *neutrum* are similar to the motives mentioned in footnote 27.

[(29)] Requirements (5) and (6) for P correspond to (1) and (2) for S. But note that no such demand is made of P as $P(a,b) = P(b,a)$, corresponding to (3) for S, nor is it stipulated that R contain a unique solution for x in $P(a,x) = b$. But such attributes may in fact happen to belong to R *without explicit requirement. If* so, then a ring R in which always $P(a,b) = P(b,a)$ is called a *commutative* ring. But since R need not thus be commutative, items (7) and (8) must separately be specified. For (7) and (8) are not balanced by requirements that $S[a,P(b,c)] = P[S(a,b),S(a,c)]$. Hence the above definition of a ring R is unsymmetrical in S and P. Any want of symmetry in the results obtained by the investigation of the logical structure of rings derives from this basic imbalance as point of origin.

theorems. Hence each ring also contains one and only one element, here called its S-*neutrum* and symbolized proleptically by the lower case Roman alphabet letter 'o', such that $S(a,o) = S(o,a) = a$ for every a in the ring. Whence it can be further demonstrated that every ring element b also verifies $P(b,o) = P(o,b) = o$.[30]

It is therefore now possible to state succinctly and clearly that retrospective analysis proves that the new *diagonal calculus*, built upon the elements and relations previously existent within the natural number system, is a ring construction. It is enough here to verify only the most delicate and decisive S-function item. The problem is therefore precisely this: given any two *diagonals* (a,A) and (b,B), is it the case that there is one and only one *unique* diagonal (x,X) such that in quasi-addition $(a,A) + (x,X) = (b,B)$? In reply note first of all that $(x,X) = (b + A, a + B)$ certainly is *one* solution, because $(a,A) + (b + A, a + B) = (a + b + A, A + a + B) = (b,B)$ since simply $a + b + A + B = A + a + B + b$. Furthermore were (y,Y) another and distinct solution, then $a + y + B = A + Y + b$, so that $y + B + a = Y + A + b$, and finally that after all $(y,Y) = (b + A, a + B) = (x,X)$.

5.7 A Comprehensive Census of the New Diagonals

Among natural numbers, and given a and A, it is the case in terms of strict alternation that

(1) either $a + 1 = A + 1 + x$ for a certain x,
(2) or $A + 1 = a + 1 + x$ for a certain x,
(3) or $a + 1 = A + 1$ altogether.

Now these three exhaustive and strictly alternative cases among the naturals take turns to yield the corresponding three alternatives among the diagonals:

(1′) either $(a,A) = (1 + x, 1)$,
(2′) or $(a,A) = (1, 1 + x)$,
(3′) or $(a,A) = (1, 1)$.

Whenever therefore x ranges over the entire domain of the naturals, the three primed formulae above deliver between them a complete census of *all* the diagonals, and each one once only, a conclusion which can be tested by reference to the completed Diagram II in section 5.3.

---

(30) Given, as above, as a group theorem verified in every R with respect to the S-function over R that $S(a,o) = S(o,a) = a$, substitute therein $a = P(b,o)$, so that $S[P(b,o),o] = P(b,o)$; then by further setting $a = o$ in $S(a,o) = S(o,a) = a$ expansion shows that $S[P(b,o),o] = P[b,S(o,o)] = S[P(b,o),P(b,o)]$ by item (7) of the conditions for a ring. Comparison moreover of the preceding extremes discloses that $S[P(b,o),o] = S[P(b,o),P(b,o)]$ which share both a common identity and a common *first* argument. Hence the *second* argument of S therein must by virtue of the uniqueness assured in item (4) of the conditions for a ring, also be common and such that $o = P(b,o)$. In similar manner it can be shown that $o = P(o,b)$.

5.8 The New Diagonals at Least as Rich as the Natural Numbers

Of these three categories of exhaustively *all* the diagonals consider now *only* the first, typified by the normal representative formula: $(1 + x, 1)$. Performed within the confines of this specified *subclass* of all diagonals, diagonal quasi-addition constructed in 5.4, and diagonal quasi-multiplication constructed in 5.5, deliver respectively the following diagonal quasi-sum (S) and diagonal quasi-product (P):

$$(S)\ (1 + x, 1) + (1 + y, 1) = (2 + x + y, 2), \text{ and}$$
$$(P)\ (1 + x, 1) \times (1 + y, 1) = (2 + x + y + xy, 2 + x + y),$$

each of which may be further simplified to read as

$$(S')\ (1 + x, 1) + (1 + y, 1) = (1 + x + y, 1), \text{ and}$$
$$(P')\ (1 + x, 1) \times (1 + y, 1) = (1 + xy, 1),$$

where all sums and products *within* parentheses are *natural* sums and products, and sums and products *between* reverse parentheses are *quasi*-sums and *quasi*-products of diagonals. Close analytical inspection of (S') and (P') will reveal that the quasi-sum between diagonals on the left side of (S') is a function of the sum between naturals within parentheses on the right side, and correspondingly that the quasi-product between diagonals on the left side of (P') is a function of the product of the naturals within parentheses on the right side. If therefore one were to exploit the connections thus transparently revealed, and pair off the diagonals of this $(1 + x, 1)$ category of diagonals against the original natural numbers x and y on the pattern that $(1 + x, 1)$ is uniquely correlated with natural x and $(1 + y, 1)$ with natural y, then thereafter without exception quasi-addition and quasi-multiplication of diagonals on the left sides of (S') and (P') would keep constant and coordinate step with ordinary addition and multiplication of naturals within parentheses on the right side of (S') and (P') respectively. *To establish such correspondence in a biunivocal manner is to recognize at one fell swoop that the natural numbers are now reconstituted as an isomorphic subdomain of the entire realm of all the diagonals.* For by the formal notion of *isomorphism* is here meant a one-to-one correspondence C between the objects and relations of a mathematical structure M and the objects and relations of a mathematical structure M', where relations of order n correspond to relations of order n, and such that whenever a relation R holds between the objects of M, the corresponding relation R' under C holds between the corresponding objects of M', and conversely. But it is here demonstrably the case that for every sum and respectively product among the domain M' of the natural numbers there corresponds one and only one quasi-sum and respectively quasi-product in the M domain of the $(1 + x, 1)$ diagonals, and conversely. The diagonal calculus therefore is *at least* as rich in resources of serial order as the domain of the natural numbers.

5.9 The New Diagonals Richer than the Natural Numbers

But if the elaborate construction of the diagonal calculus were *at most only* as rich in resources of order as the domain of the natural numbers, then would the fraternity of professional mathematicians merit perhaps the opprobrium of the populace for having much to do about nothing in their search for a legitimized zero. But it is happily demonstrably the case that the diagonal calculus is incalculably *richer* than the domain of the natural numbers. For the careful reader may now recall in a flash of insight that in footnote 21 to section 4.1 wherein the set S was interpreted as the set of natural numbers, certain attributes of the natural numbers were listed and in particular item (4) to the effect that within the domain of the naturals $a + x = b$ *but not always solvable in the set*. For the serial order of the domain of natural numbers is severely limited. In particular the equation $2 + x = 1$ is among the natural numbers frankly *unsolvable*.[31] But if one now substitutes for the *natural number* coefficients in this equation their addition-true and multiplication-true *diagonal isomorphs*, so that $2 + x = 1$ is transposed to read $(3,1) + (x,X) = (2,1)$, a unique solution exists and is readily identifiable as the diagonal $(x,X) = (1,2)$ because $(3,1) + (1,2) = (4,3) = (2,1)$ which on the isomorphic pattern of $(1 + x, 1)$ to x, corresponds to 1.[32] Where then the resources of serial order fail to keep pace within the natural number domain with the fertility of the relations constructed over it, there is no alternative but to enter the realm of diagonals and therein subtract the diagonal isomorph of natural 2 from the diagonal isomorph of natural 1. This result really does exist in the realm of diagonals, *but it is a diagonal which is not the isomorph of any natural number*.

6.0 The Diagonals As Positive and Negative Integers

But a diagonal, as Gertrude Stein or even Aristotle would remind us, is a diagonal is a diagonal. Hence if the diagonal (2,1) is a number, so too is the fellow-diagonal (1,2) likewise a number. Now therefore that the process of rigorous construction is happily completed, and there is little danger any longer that familiarity with symbols in computation may breed logical slovenliness with respect to their intellectual justification, we may safely revert to conventional practice and regard the diagonal typified by $(1 + x, 1)$ as

---

[31] Nor should the reader allow a superficial and deceptive analogy between the new (a,A) and the old $a - A$ tempt him thoughtlessly to put $x = (1,2)$. Such procedure is sheer nonsense insofar as a statement like '$2 + (1,2) = 1$' attempts to add elements so disparate as an individual natural number and an individual *class* of *couples* of natural numbers.

[32] It is *not* true that what mathematical convention knows as the negatively signed integer —1, results from subtracting the natural number 2 from the natural number 1. This is a pseudo-process that cannot be performed nor even its idea countenanced by an honest mathematical conscience.

synonymous with $+ x$,[33] and correspondingly the diagonal typified by $(1, 1 + x)$ as synonymous with $-x$. At this point therefore let us revert to the correctly completed Diagram II in section 5.3 and inscribe thereon as labels at the top of each but the first vertical column and from left to right the corresponding negatively signed integers on the representative pattern $(1, 1 + x)$ to $-x$, and in the sequence therefore -1, -2, -3, -4, etc., and similarly at the left of each but the first horizontal row in descending order from top to bottom the corresponding positively signed integers on the representative pattern $(1 + x, 1)$ to $+ x$, and in the sequence $+ 1$, $+ 2$, $+ 3$, $+ 4$, etc.[34]

6.1 Now Nothing Remains or Nought Is Left but Zero

Fixation of reflective attention upon the completed and now significantly ornamented Diagram II of section 5.3 will disclose that the serial order riches of the diagonal calculus, already far superior to those of the natural numbers, are not yet exhausted. For there remains solitary and unique among all the diagonals the central one, distinguished from all of the others above and below by the mark that both of the components of each of its couples is identically the same natural number. But although different and distinct, this central diagonal is nevertheless a diagonal among its diagonal peers, sharing without compromise or exception exactly all of their mathematical theory, equally enfranchised by rigorous construction and equally existent, and—philosophers please note—completely innocent of all reference to counting where there is nothing to count. It is the diagonal typified by (1,1), symbolized by 0, and called *not* nothing *nor* nought but the *number* zero.[35]

---

[33] *Not* with natural number x of course. Whoever does not now see the reasons for this prohibition, should begin again to con this essay from its first pages.

[34] Here it is perhaps opportune to redeem the pledge provoked by Clavius' mournful challenge in section 4.2 to transform the status of the Rule-of-Signs from a mere gentlemen's agreement that just happens to work well in computational experience to that of a *demonstrated theorem*. Such proof may perhaps most briefly be conveyed by carrying out the process of a simple multiplication in the diagonal calculus: $(1,2)(1,2) = (5,4)$ by the definition of class product among diagonals, and $[5,4] \equiv [2,1]$ by the definition of classmateship, and the normal representative (2,1) is isomorph to 1. The Rule-of-Signs thus becomes a *demonstrated theorem* because (1,2) is synonymous with -1 after the normal representative pattern of correspondence of $(1, 1 + x)$ to $-x$, and (2,1) is synonymous with $+ 1$ after the normal representative pattern of correspondence of $(1 + x, 1)$ to $+ x$. But two further observations are relevant: (a) one cannot however write $(1,2)(1,2) = 1$ because isomorphism is *not* identity of individuals, but one may of course conveniently agree to write elliptically for economical reasons 1 instead of $+ 1$; (b) the Rule-of-Signs thus becomes a theorem but *only* by the grace of the principle of isomorphic subdomain.

[35] Three final comments are perhaps in order here: (1) note in general that there is no *direct* contact whatever between the domain of the natural numbers and the realm of the general integer classes, but indirect contact is won by the fact that the natural numbers are *isomorph* to the *positive* integers which are themselves *homogeneous* with the *general* integer classes; (2) *order* among the integer classes

must also be *contructed* abstractly and independently; finally (3) rich in serial order as is the system of the positive, null, and negative integers when thus constructed in their capacity as a difference ring over the natural numbers, the integer domain is not yet rich enough to take care of the logical fertility of the relations established over it and therefore requires extension by further rationally motivated, critically executed, and autonomous construction of the realm of the rationals, the irrationals, or the real and complex number domains. But in each case the procedure is similar and proceeds according to the principle of isomorphic subdomain. I close with a selected bibliography on the topics of this essay which pays attention to details of historical development: [1] E. Study, "Theorie der gemeinen und höheren komplexen Zahlen," *Enzyklopädie der mathematischen Wissenschaften* I. 1(1899). 2; [2] K. Th. Vahlen, *Abstrakte Geometrie* (Leipzig: Teubner, 1905), I, pp. 7-52; [3] E. Steinitz, "Algebraische Theorie der Körper," *Crelle* 137 (1910) 167-308; [4] O. Stolz and J. A. Gmeiner, *Theoretische Arithmetik* (Leipzig: Teubner) 1: rationale Zahlen^2 (1911), 2: reelle und komplexe Zahlen^2 (1915); [5] H. Wieleitner, *Die sieben Rechnungsarten mit allgemeinen Zahlen* (Leipzig: Teubner, 1912); [6] O. Hoelder, *Die Arithmetik in strenger Begründung* (Leipzig: Teubner, 1914); [7] A. Loewy, "Die Grundlagen der Arithmetik," *Lehrbuch der Algebra* (Leipzig: Veit, 1915); [8] H. Wieleitner, *Der Begriff der Zahl in seiner logischen und historischen Entwicklung*^{2} (Leipzig: Teubner, 1918); [9] O. Perron, *Irrationalzahlen* (Berlin and Leipzig: de Gruyter, 1921); [10] A. Pringsheim, *Vorlesungen über Zahlen- und Funktionenlehre* (Leipzig: Teubner) 1.1^2 (1923), 1.3 (1921); [11] E. Landau, *Grundlagen der Analysis* (Leipzig: Akademische Verlagsgesellschaft, 1930), reprinted in German and separately issued also in an English translation by Chelsea Publishing Co., New York, N. Y., 1951; [12] Roy Dubisch, *The Nature of Number* (New York: The Ronald Press, 1952).

Astronomy

SOME JESUIT CONTRIBUTIONS TO ASTRONOMY

HENRY M. BROCK, S.J.

To an ordinary person the title of this paper may seem a little incongruous. For the mention of our Society conjures up for him the memories of our Saints, our great missionaries, theologians, preachers and teachers who in its early years helped to stem the tide of the so-called Reformation in Europe and who since then have done so much to spread the Kingdom of Christ throughout the world. If some one should ask what have clerics to do with a science like astronomy, we might perhaps cite the statement of Fr. Koch in his *Jesuiten Lexikon*: "Astronomy, the oldest science, the daughter of religion, has always been a favorite of the Catholic Church." In ancient times it determined the calendar days for planting and for celebrating religious festivals, etc. As far back as the year 325 A.D., the Council of Nicea concerned itself with the positions and phases of the moon in determining our Easter Cycle. The great work of Copernicus (d. 1543) expounding the heliocentric theory was dedicated to Paul III. In 1552, Gregory XIII promulgated the reformed calendar bearing his name which we now follow. In modern times the Holy See has for many years maintained its own observatory. Pius XI once declared to a group of astronomers: "The science that you profess is the noblest and most beautiful that exists." He showed his practical interest by moving the Vatican Observatory from Rome to Castel Gandolfo and equipping it with the most modern instruments. Only last September Pius XII welcomed the members of the International Union of Astronomers at his Observatory, met them individually and gave them a masterly address (*A.A.S.*, Oct. 16, 1952: *Cath. Mind*, Jan. 1953).

It is not surprising then that there has been a traditional interest and devotion to science, and especially to astronomy, in the Society which may go back to the days of St. Ignatius. Among the Saints he is known for his love of the starry heavens. We remember his saying: "Quam sordet mihi tellus dum coelum aspicio." For a Jesuit astronomy is not merely one of the many human sciences. He sees in it a distinctly apologetic value manifesting as it does the power of the human mind and the omnipotence and infinite intelligence of God. He knows, too, how our observatories in mission lands have gained prestige for the Church and the missionaries, as well as good will and often favors from civil rulers. We propose to make a brief survey of the Jesuit observatories and of some of the achievements of our Fathers in the science of the Heavens.

OBSERVATORIES.

It is rather surprising to note the number of observatories established in both the old and new Society. Probably the first in the old Society and one of the larger ones was the one in Pekin, founded by the Emperor of China in 1668 and placed in the charge of our Fathers. This was about the time that the National Observatories at Greenwich and Paris were established. The Vienna National Observatory founded in 1745 was also conducted by the Jesuits. We also read of smaller institutions usually attached to a College in Marseilles, Lisbon, Vilna, Prague, Milan, Florence, Rome, Parma etc. Obviously we would not expect any in the countries where the new religion had gained the ascendency. It is not likely that any of these were recovered after the Restoration. However as the Society grew in numbers and began again the work of education, we meet a number of more familiar names: Stonyhurst in England, Kalosca in Hungary, Tortosa in Spain, Valkenburg in Holland, Louvain in Belgium, Jersey (one of the Channel Islands), Ksara in Syria, Zi-Ka-Wei in China, Manila, Riverview in Australia, Tananarivo in Madagascar etc. In South America there are two small observatories attached to Scholasticates, one near Rio de Janiero, the other near Buenos Aires. In the United States we have Georgetown, Crieghton, Santa Clara, Woodstock, Weston and Holy Cross. Most of these are rather small and intended chiefly for instruction. Tortosa and Zi-Ka-Wei have sections devoted to Seismology and Terrestrial Magnetism. Manila also had Meteorology and Seismology. The Vatican Observatory belongs to the Holy See but it has a Jesuit Staff, the new Director being Fr. Daniel O'Connell; formerly of Riverview. The present state of some of these observatories is significant. Kalosca has been lost. The Jersey community has gone back to France. Valkenburg was taken over by the Nazis. Manila was completely destroyed in the war, but a new beginning has been made at Baguio. Zi-Ka-Wei is now in the hands of the Chinese Communists. Stonyhurst has been closed indefinitely because of the lack of men and resources.

MOON:

We are naturally more interested in the astronomical achievements of our Fathers than in the instrumental equipment employed, though this is also worthy of note. We may consider now some of the fields in which they have labored. We may begin with our nearest neighbor in space. Galileo was the first to view the moon with a telescope. While Hevel of Danzig was the first to attempt a lunar map, our present map and its nomenclature are due mostly to Fr. Riccioli (d. 1671), who took up astronomy after teaching Philosophy and Theology for twenty years. He was aided by Fr. Grimaldi

(d. 1663), the discoverer of Diffraction in Optics. Fr. Grimaldi collaborated with Fr. Riccioli in the publication of the latter's great work: *Almagestum Novum* and supplied the lunar map in Vol. I. They named many of the principal craters after eminent philosophers and men of science. If we examine the surface of the moon and check the places observed on a lunar map, we may be surprised to find two adjacent craters near the eastern limb bearing the names "Riccioli" and "Grimaldi." We may ask how they got there. Two astronomers, Littrow and Madler declared in no friendly spirit that Fr. Riccioli was responsible for this. Fr. Koch seems to admit it as an example of "Gelehrteneitelkeit" or pedantic vanity which did no harm. Certainly today no one would deny this honor to these two men in view of their achievements in physics and astronomy. Among other Jesuit names on the moon-map we may mention Frs. Clavius, Hell, Triesnecker, Secchi, deVico. Riccioli also studied the lunar libration. Modern lunar research has centered chiefly on the occultation of stars by the moon. Excellent work of this kind has been done at Georgetown and Weston, some of it in response to requests for data to check the new Brown lunar tables.

SUN:

Solar research has always been favored by the Jesuit astronomers. The pioneer in this important field was Fr. Christopher Scheiner (d. 1650). He constructed his own telescope and also conceived the idea of projecting the sun's image on a screen in order to study its surface. He was an independent discoverer of sunspots. Though Galileo claimed priority on this point, Scheiner was the first to follow up his discovery by continuous observations over many years. He noted especially the faculae and the motion of the spots and determined the period of the rotation of the sun. In the 19th Century we meet among others the names of Angelo Secchi (d. 1878) at Rome, Fenyi at Kalosca, and Stephen Perry (d. 1889) at Stonyhurst. They observed the spots and faculae. Secchi's book, *Le Soleil* was a standard work for many years. It includes also the best spectrum maps available at the time. At Tortosa the sun is photographed regularly. The magnetic station at Stonyhurst, in charge of Fr. Perry, contributed much to establishing the relationship between the sunspot frequency and magnetic storms.

SOLAR ECLIPSES:

A total eclipse of the sun is a striking, and in any given place, a rather rare phenomenon and we are not surprised to learn that our astronomers do not hesitate to travel long distances with their equipment to observe one. They may be following an old tradition for we are told that Bl. Charles Spinola was one of those who observed the eclipse of Nov. 8, 1692 in Japan. The important eclipse of 1860 was observed in Spain by Fr. Secchi and his results proved conclusively that the prominences are solar in origin and that they are found all the way around the sun. He and de la Rue, an English astronomer, at a nearby station, used photography for the first time successfully at an eclipse. Fr. Perry had the distinction of having been placed in charge of four eclipse expeditions by the British Government, a naval vessel being placed at his disposal each time. Thus his travels took him to Spain in 1870, to the West Indies in 1886, to Russia in 1887, and to the Palut Islands off South America in 1889. While at the last place, he contracted a fever and died at sea far from his brethren five days after the eclipse. Mitchell in his book *Eclipses of the Sun* says of him:

> The eclipse of Dec. 22, 1889 is memorable for the death of Father Perry a few days after the eclipse, a martyr to the cause of science. This brave man, though greatly weakened, took part in the eclipse work and having found as soon as totality passed that everything had turned out well, he called for three hearty British cheers which unfortunately he himself could not lead.

We may add that Fr. Perry was also a pioneer in photographing the spectra of sunspots.

During the past twenty years Georgetown has become well known for its successful eclipse expeditions. At Freyburg, Maine on Aug. 31, 1932 the Jesuit group under Fr. Paul A. McNally, with only modest resources, obtained excellent photographs of the corona and prominences. One of the former is reproduced on p. 299 of Baker's *Astronomy* (5th Ed. 1950). Their success led to invitations from the National Geographic Society and afterwards from the U. S. Armed Forces to participate in expeditions made to Russia in 1936, to Patos, Brazil in 1938, to Canton Island, So. Pacific in 1940, to Bocaiuva, Brazil in 1947, to Wuchang, China in 1948 and to Khartoum in Africa in 1952. The last three expeditions were under the direction of the present Director, Father Francis J. Heyden. It may be of interest to note that the reports of the partial eclipse of Jan. 24, 1925, observed by a group here at Weston and by Fr. Edward Phillips at Woodstock, were published in *Popular Astronomy* the following April. Interesting photographs were obtained here.

PLANETS:

For a long time observations of the transits of the planet Venus across the sun's disc at two widely separated stations were about the best means of determining the solar parallax, from which the distance of the earth to the sun is obtained. This does not occur often. Fr. Maximilian Hell (d. 1792) by invitation of the King of Denmark observed the transit in 1769 in Norway. Fr. Perry was sent out to observe the transit of 1874 at Kerguelen in the Indian Ocean and that of 1882 in Madagascar. As far more precise methods are now available it is doubtful if any Jesuit expedition will be sent to observe the next transit of Venus in 2004.

COMETS:

Among comet seekers the name of Fr. Francesco de Vico (d. 1848) is conspicuous. Under him the Observatory of the Roman College acquired a European reputation. His greatest achievement was the discovery of eight comets, one of which is named after him. For this he received the prize of the French Academy and six gold medals from the King of Denmark. One of the latter is now at Georgetown. He came to the University in 1848 on account of political disturbances in Rome and was appointed Director of the Observatory. He went to England shortly afterwards to purchase equipment and died of typhus in Liverpool. The most recent of the comets discovered by a Jesuit is Comet Timmers, named for the Brother Mechanic of the Vatican Observatory, Bro. Matthew Timmers, who discovered this object on Feb. 2, 1945 while examining photographic plates taken at Castel Gandolfo. For his discovery, Brother Timmers received the Donahue Comet Medal of the Astronomical Society of the Pacific.

STARS:

Like their colleagues, Jesuit astronomers have carried out routine stellar observations. Thus, for example, the 37 Volumes of the *Ephemerides Astronomicae* published at the Vienna Observatory under the direction of Fr. Maximilian Hell give evidence of this kind of work. In this immense field, Fr. Secchi again merits special attention as one of the founders of stellar spectroscopy. When the principles of spectrum analysis had been enunciated and tested by Bunsen and Kirchoff about the middle of the 19th century, Sechi and the English astronomer Huggins were the first to realize practically their importance in opening up entirely new fields of research in the study of the composition and nature of the sun and stars. Huggins studied the spectra of relatively few stars in some detail, while Secchi on the other hand undertook a more necessary and timely task involving much skill and labor, viz.: a spectroscopic survey of the heavens. From 1863 to 1867 he examined the spectra of some 4000 stars by visual observation. A study of his results revealed to him the fact that they could be grouped into four classes, a discovery of great importance. The latter served a useful purpose for many years until superseded by the more detailed Draper Classification.

In our day the study of variable stars has become important and in this field Fr. John G. Hagen (d. 1930) gained eminence. Coming to America in 1880 he taught mathematics at the House of Philosophy of the Buffalo Mission at Prairie du Chien. Here he took up astronomy and began the study of variable stars. He became the Director of the Georgetown Observatory in 1888 and added much to its reputation. He made many improvements, installing among other things the present 12″ refractor. While there he published his *Synopsis der Höheren Mathematik* (3 Vol.). He was an assiduous observer of the then known variables which were comparatively few in number. He also published a large number of beautifully engraved charts to aid other astronomers. Bl. Pius X appointed him Director of the Vatican Observatory in 1906. Here he published his *Die Veränderlichen Sterne* (2 Vol.) with Fr. J. Stein who later became his successor. Fr. W. Miller of the Vatican Observatory has published numerous papers on variable stars. His latest work on the "Vatican Variables" contains many suggestions to observers in this field regarding techniques and the control of data. Fr. F. Heyden of Georgetown is a member of the I.A.U. Commission on Variable Stars. His researches on the colors of the Cepheid Variables demonstrated the value of this method of determining the effects of interstellar reddening at great distances from the sun. Fr. O'Connell when at Riverview and Fr. Depperman at Manila have also done excellent work on Variables.

THE CALENDAR:

Fr. Christopher Clavius (d. 1612), the "Euclid of the 16th Century" is best known for having worked out the corrections and reform of the old calendar which had been in use since the days of Julius Caesar. It was with his advice that Gregory XIII, as had already been stated, promulgated the new calendar in 1582. He wrote in its defense, but on account of its papal origin non-Catholic countries were slow in adopting it. Thus England, including the American colonies, waited 170 years and most of the Orthodox countries until after the first World War.

TIME AND LONGITUDE:

One of the early useful applications of radio was the broadcasting of time. On account of its increasing importance this was done usually only by Government institutions as for example, the U.S. Naval Observatory and the U.S. Bureau of Standards. So it is interesting to note that this public service in the Far East was performed for many years before the last war at Manila and Zi-Ka-Wei. The clocks used were checked regularly by the stars.

During October and November of 1926 an International Longitude Operation was carried out with the cooperation of astronomers throughout the world. Its purpose was to determine the difference in longitude of a number of observatories in order to test the permanency of their relative positions and thus to explore certain possibilities as to the movements of the earth's crust and also to enable various observatories to check on their longitudes. Three key stations with positions accurately known were chosen in approximately the same latitude about 8 hours apart in longitude. They were charged with sending out time signals at stated intervals. One of these as Zi-Ka-Wei where the work was carried out with success under Fr. Paul Lejay. The other two were San Diego, California and Algiers. On this occasion Fr. Phillips at Georgetown and Fr. Depperman at Manila redetermined the longitudes of their observatories.

Chronology and Babylonian Astronomy:

In the course of extensive excavations at Babylon during the 19th century thousands of tablets were found with cuneiform inscriptions. Many of these came to the British Museum. Oriental scholars were able to decipher many of them, but the astronomical records remained a mystery. This was solved by Fr. Joseph Epping (d. 1884), an astronomer, and Fr. N. Strassmaier (d. 1920), an Assyriologist, working together—an ideal combination. After much labor the key was found and the various symbols for the planets were identified and and the tables were interpreted. Thus the remarkable astronomical knowledge of the Babylonians was made known to the world. The *Encyclopedia Britannica* (Vol. II, p. 580) says

> The decipherment and interpretation by the learned Jesuit Fathers Epping and Strassmaier of a number of clay tablets preserved in the British Museum, has supplied detailed knowledge of the methods practiced in Mesopotamia in the 2nd Century B.C.

Father F. X. Kugler (d. 1925) continued their work with brilliant success. He established some important dates. Thus the discovery of the cuneiform table of the movements of Venus opened the way for the solution of the vexing question of the date of Hammurabi. As a result we may say with great probability that his reign falls between 1728 and 1686 B.C. Both Epping and Kugler published learned works in their field.

Inventions:

Astronomers are not usually inventors. Their instruments and accessories are usually designed and constructed by experts in other fields. A few Jesuit inventors may however be mentioned. The principle of the telescope was discovered in Holland and Newton constructed the first reflecting telescope. Galileo made several refracting telescopes with a concave lens eyepiece. Fr. Scheiner however was the first to design and construct a refractor along modern lines after Kepler pointed out the advantage of a convex eyepiece lens. It had chromatic and also, no doubt, spherical aberration but he got good results with it. He also invented the pantograph. Clavius discovered the method of subdividing the smallest division of a scale. Vernier used it in the familiar attachment to measuring instruments which has his name. Fr. R. Boscovich (d. 1787) invented the ring micrometer. Among the Jesuit "firsts" should be mentioned the development by Fr. Secchi of a new type of spectroscope using a direct vision prism. Fr. G. Fargis (d. 1916) constructed at Georgetown his photochronograph to record star transits on a photographic plate. It did not come into use at the time but the method is now used with improvements at the U.S. Naval Observatory in checking the clocks that give us our time.

Fr. W. Rigge (d. 1927) of Creighton, author of a book on the

Geographical Construction of Eclipses and Occultations designed and constructed a remarkable machine for drawing complex curves by a combination of several periodic motions. It is described in his book *Harmonic Curves*.

When star transits are observed a correction must be applied to the observed time on account of the instrumental errors. This is equal to the sum of three products. The process is rather tedious. Fr. E. Phillips (d. 1951), former Director of the Georgetown Observatory, designed and constructed what he called a "Transit Reduction Computing Machine" by means of which the correction can be obtained mechanically without calculation. The astronomers of the U.S. Naval Observatory were much impressed by it.

As is clear, this survey makes no claim to be complete. Time only permits mention of the contributions of astronomy in aiding the labors of the missionaries of the old Society in China. This is well described in the works of Ricci, Verbiest and Schall. The Jesuit ambassadors of Christ to China were faced with a mighty problem: how to penetrate the huge isolated country that was China with its traditional hatred of all things foreign. These men realized that no direct frontal attack would be possible. They entered as scholars, bringing with them books and instruments from Europe and thus aroused the attention and interest of the Chinese astronomers. The latter, marvelling at the proficiency of these men from the West in astronomical learning, saw to it that word reached the Emperor. He, in his turn, asked them for new instruments of European design. Fr. Verbiest planned an observatory and designed and constructed certain instruments which remain to this day. It was in this way that the Jesuit scientists of the 17th century gained favor for their fellow missionaries to preach the good-news of Christ in the Kingdom of China. Certain reports went to Rome criticizing the methods of these missionary scholars. However they were consoled by receiving a letter from the Pontiff, Innocent XI, commending them for using the profane sciences in working for the salvation of the Chinese people.

Chemistry

THE DISPROPORTIONATION OF PRIMARY ALIPHATIC AMINES IN THE PRESENCE OF RANEY NICKEL CATALYST[1]

JOSEPH A. MARTUS, S.J.

The catalytic hydrogenation of nitriles in the presence of certain finely divided metals, such as nickel or platinum, has generally resulted not only in the desired yield of primary amine, but also in an unwelcome yield of secondary amine. In this work the disproportionation of primary amines was studied with the purpose of discovering those conditions under which the largest yields of secondary amines can be produced.

The disproportionation of the following amines was investigated: aniline in the presence of benzylamine, m-nitro-aniline in a solution of dipentene, aniline in a solution of tetralin, p-toluidine in a solution of dipentene, benzylamine without a solvent, benzylamine in a solution of xylene, cyclohexylamine without a solvent, and the following amines, also without a solvent: decylamine, nonylamine, octylamine, n-amylamine, n-butylamine.

The disproportionation of a primary aliphatic amine requires the presence of an active catalyst, and in this work Raney nickel produced the best results. The ratio of amine to Raney nickel catalyst on a gram basis was roughly two to one. In one portion of the work activated alumina was employed as a catalyst, but it did not prove as successful as Raney nickel.

A variety of solvents was at first employed, to lessen the formation of tars. It was found however, during the course of the investigation that no solvent need be used, if the reaction were conducted in an atmosphere of hydrogen gas. For the sake of comparison one experiment was conducted wherein octylamine underwent disproportionation in the presence of oxygen gas. The yield of di-octylamine was considerably reduced. The use of nitrogen gas was not as helpful as hydrogen gas.

Some of the reactions were performed at room temperature without any disproportionation taking place. These reactions were performed solely in the presence of activated alumina. The remaining reactions were conducted at an elevated temperature, generally in the range of 190°-210°C. At this temperature large yields of secondary

[1] Abstract of a dissertation submitted to the Faculty of Clark University, in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Chemistry.

amines were obtained. For the most part the reactions were allowed to run from four to five hours.

The results of the investigation show that the disproportionation of primary aliphatic amines proceeds only with great difficulty. There is required an active catalyst, such as Raney nickel, and a high temperature. Under the conditions investigated the aryl amines gave no detectable yield of secondary amine. Cyclohexylamine, under the given conditions, likewise gave an extremely small yield of secondary amine. Benzylamine, in the absence of a solvent and in an atmosphere of hydrogen gas, gave a 51% yield of di-benzylamine. When the hydrogen was passed over the surface of the liquid, benzylamine gave a 62% yield of secondary amine. Decylamine in the presence of hydrogen gas and heated for 1.25 hours gave a 32% yield of didecylamine. Nonylamine in the presence of hydrogen gas and heated for 2 hours gave a 24% yield of di-nonylamine. Octylamine, heated for four hours in the presence of hydrogen gas, gave a 76% yield of dioctylamine. n-Butylamine, heated in the presence of hydrogen gas in an autoclave, gave a 50% yield of di-n-butylamine. Octylamine, heated in the presence of oxygen gas, gave a 48% yield of di-octylamine.

Reviews and Abstracts

General Education In Science, edited by I. Bernard Cohen and Fletcher G. Watson. Harvard University Press, xviii + 217 pp. 1952.

"Why should we teach any science at all to the nonscientists?" is the blunt query in an opening address at the 1950 Workshop Concerning Science in General Education. The responses of the other fourteen guiding contributors may be gleaned from their addresses as presented in this volume. The question is a burning one—and the response no less so, since the why and wherefore are nothing but a statement of *objectives* which in turn determine the *means* to be employed, i.e., content of course and mode of presentation.

This query is followed by an analysis of the objectives of general education itself, with a terse summary of it as "the improvement of citizen understanding and clearer thinking." A second query presses the attack, "Is there anything *unique in science* that will contribute to better citizen understanding and clearer thinking?" The technique of solving problems is the substance of one reply. Another is the insight that science is a force not only for social *change* but also for social *stability*, with the consequent importance of an historical perspective, of realizing the interaction of science with religion, literature, and all other human activity. Reference is made to readings which probe these to-and-fro influences in history. Examples are cited—Ben Franklin and Asa Gray, in contrast to Sam Johnson and Louis Agassiz—as evidencing this *integral* view when faced with decisions in matters of great complexity.

As a response to the same question, a remedy for the same ailment, the case history method of teaching science is discussed by those who have been attempting it. Shortcomings and obstacles are brought to light. The average teacher's dearth of an historical background is recognized, and specific books are suggested to those desirous of acquiring this perspective. Teachers' conferences have proven helpful. Another difficulty is broached: how to give the student true practise in clear thinking? i.e., in problems the solutions to which are unknown to him beforehand. Colgate attempts an answer; the results of a student questionnaire are cited as evidence of its efficacy. Then another thorn: how *evaluate* student progress? Essay-type tests are declared indispensable, although one contributor presents a sample set of objective questions designed to achieve this goal.

Two speakers consider the relationship of the philosophy of science to the teaching of science, discussing the importance of a scientist's environment,—of the time-space setting in which he works, with its tremendous influence on his "objectively-determined" hypotheses and theories. The role of *applications* in teaching science is presented with insight and appreciation. Some problems in the teaching of biology are diagnosed by another pair of contributors, who iterate the importance of natural history and of laboratory work. The two concluding papers present techniques of evaluation, as applied to the case history method of teaching.

This book will interest most science teachers—fresh, relevant ideas are always welcome, and this workshop provides more than a trickle. As a workshop too, it is a bit jarring. Here are men from Harvard, Colgate, and Yale, from Brookhaven and the Rockefeller Institute for Medical Research taking time out to reflect together on the needs of our day, on their objectives in teaching science, and on the means of achieving them. We tend to take our objectives and means,—the courses, as set in concrete. This is due in part to our inheritance of centuries of achievement. Yet if it is due at all to the effectiveness of what we have, perhaps we could illuminate those who are still groping. That too is part of our purpose. Moreover, those centuries of Jesuit experience and achievement in the classroom were stamped by what Gilbert Highet, no mean teacher himself, terms "adaptation." Ultimate objectives remain fixed, but the multitude of intermediate objectives and means,—are they by nature unvarying? The book is worth reading.